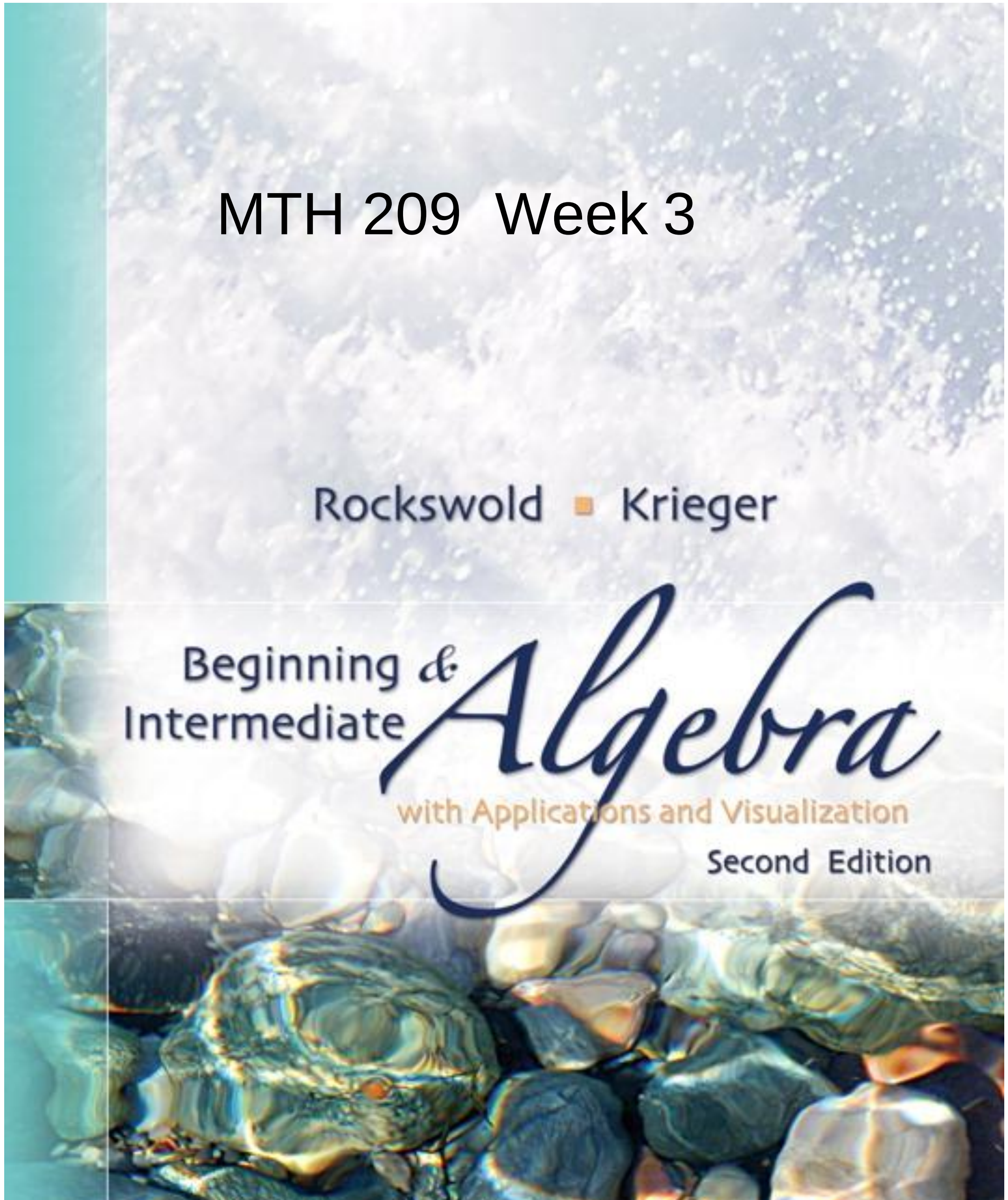
MTH 209 Week 3



Due for this week…

- Homework 3 (on MyMathLab – via the Materials Link) → Monday night at 6pm.
- Read Chapter 8.4, 10.4, 11.1, 11.5,12.1-3,14.1-3
- Do the MyMathLab Self-Check for week 2.
- Learning team planning for week 5.
- Discuss your final week topic for your team presentations…

6.6 Solving Equations by Factoring I (Quadratics)

- The Zero-Product Property
- Solving Quadratic Equations
- Applications

To solve equations we often use the **zero-product property,** which states that if the product of two numbers is 0, then at least one of the numbers must be 0.

ZERO-PRODUCT PROPERTY

For all real numbers a and b, if $ab = 0$, then $a = 0$ or $b = 0$ (or both).

EXAMPLE Applying the zero-product property

Solve each equation.

a. $3x(x+4)=0$

b. $(4x-1)(3x+4)=0$

Solution

a. $3x(x+4)=0$

$3x(x+4)=0$

$3x=0 \quad \text{or} \quad x+4=0$

$x=0 \quad \text{or} \quad x=-4$

b. $(4x-1)(3x+4)=0$

$(4x-1)(3x+4)=0$

$4x-1=0 \quad \text{or} \quad 3x+4=0$

$4x=1 \quad \text{or} \quad 3x=-4$

$x=\frac{1}{4} \quad \text{or} \quad x=\frac{-4}{3}$

Any **quadratic polynomial** in the variable x can be written as $ax^2 + bx + c$ with $a \neq 0$.

Any **quadratic equation** in the variable x can be written as $ax^2 + bx + c = 0$, with $a \neq 0$. This form of quadratic equation is called the **standard form** of a quadratic equation.

SOLVING QUADRATIC EQUATIONS

To solve a quadratic equation by factoring, follow these steps.

STEP 1: If necessary, write the equation in standard form as $ax^2 + bx + c = 0$.

STEP 2: Factor the left side of the equation using any method.

STEP 3: Apply the zero-product property.

STEP 4: Solve each of the resulting equations. Check any solutions.

EXAMPLE Solving equations by factoring

Solve each quadratic equation. Check your answers.

a. $36 + x^2 = -12x$

b. $x^2 - 49 = 0$

Solution

a. $36 + x^2 = -12x$

$$x^2 + 12x + 36 = 0$$

$$(x + 6)(x + 6) = 0$$

$$x = -6$$

To check this value, substitute −6 for x in the given equation.

$$36 + (-6)^2 = -12(-6)$$

$$72 = 72$$

The only solution is −6.

EXAMPLE continued

Solve each quadratic equation. Check your answers.

b. $x^2 - 49 = 0$

Solution

b. $x^2 - 49 = 0$

$(x + 7)(x - 7) = 0$

$x + 7 = 0 \qquad x - 7 = 0$

$x = -7 \qquad x = 7$

To check these values, substitute 7 and −7 for x in the given equation.

$7^2 - 49 = 0 \qquad (-7)^2 - 49 = 0$

$0 = 0 \qquad 0 = 0$

The solutions are −7 and 7.

EXAMPLE Solving an equation by factoring

Solve $2x^2 - 7x = -5$

Solution

$$2x^2 - 7x = -5$$

$$2x^2 - 7x + 5 = 0$$

$$(2x - 5)(x - 1) = 0$$

$$2x - 5 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = \frac{5}{2} \quad \text{or} \quad x = 1$$

The solutions are 1 and $\frac{5}{2}$.

EXAMPLE Modeling the flight of a model rocket

If a model rocket is launched at 48 feet per second, then its height, h, after t seconds is $h = 48t - 16t^2$. After how long does the rocket strike the ground?

Solution

The rocket strikes the ground when the height is 0.

$$48t - 16t^2 = 0$$

$$16t(3 - t) = 0$$

$$16t = 0 \qquad 3 - t = 0$$

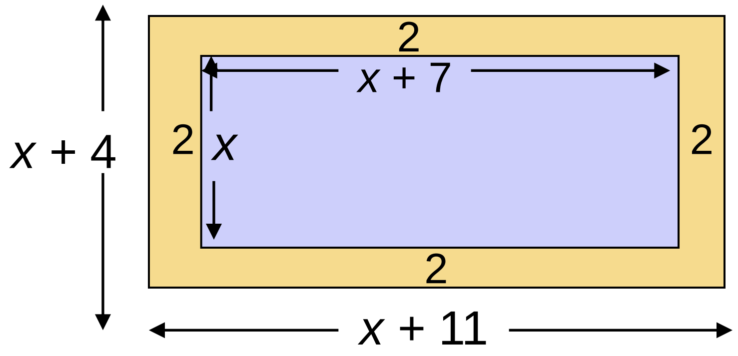
$$t = 0 \qquad t = 3$$

The rocket strikes the ground after 3 seconds.

EXAMPLE Finding the dimensions of a picture

A frame surrounding a picture is 2 inches wide. The picture inside the frame is 7 inches longer than it is wide. If the overall area of the picture and frame is 198 square inches, find the dimensions of the picture inside the frame.

Solution Let x be the width of the picture and $x + 7$ be its length.



EXAMPLE continued

$$(x + 4)(x + 11) = 198$$

$$x^2 + 15x + 44 = 198$$

$$x^2 + 15x - 154 = 0$$

$$(x - 7)(x + 22) = 0$$

$$x - 7 = 0 \quad \text{or} \quad x + 22 = 0$$

$$x = 7 \quad \text{or} \quad x = -22$$

The only valid solution for x is 7 inches. Because the length is 7 inches more than the width, the dimensions are 7 inches and 14 inches.

Practice for section 6.6

- Applying the zero-product property Q 11-21
- Solving equations by factoring Q 23-48
- Solving an equation by factoring Q49-58
- Modeling the flight of a model rocket Q 65a
- Finding the dimensions of a picture Q 71

6.7 Solving Equations by Factoring II (Higher Degree)

- Polynomials Having Common Factors
- Special Types of Polynomials

Polynomials Having Common Factors

The first step in factoring a polynomial is to factor out the greatest common factor (GCF).

$$x^3 - x = x(x^2 - 1)$$
$$= x(x - 1)(x + 1)$$

EXAMPLE Factoring trinomials with common factors

Factor the trinomial completely. $6x^3 - 21x^2 - 12x$

Solution

Start by factoring out $3x$.

$$3x(2x^2 - 7x - 4)$$

Factor the trinomial. $3x(2x + 1)(x - 4)$

EXAMPLE Solving polynomial equations

Solve the equation. $x^3 - x^2 - 12x = 0$

Solution

$$x^3 - x^2 - 12x = 0$$

$$x(x^2 - x - 12) = 0$$

$$x(x - 4)(x + 3) = 0$$

$$x = 0 \quad \text{or} \quad x - 4 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 0 \text{ or } x = 4 \text{ or } x = -3$$

Special Types of Polynomials

Some types of polynomials of higher degree can be factored by using methods that we have already presented.

EXAMPLE Factoring higher degree polynomials

Factor each polynomial completely.

a. $x^4 - 25$ b. $y^4 + 5y^2 + 6$

Solution

a. Difference of squares. $x^4 - 25$

$x^4 = (x^2)^2$

$= (x^2)^2 - 5^2$

$= (x^2 - 5)(x^2 + 5)$

b. Because $y^2 + 5y + 6 = (y + 2)(y + 3)$, we let $a = y^2$ and then factor the given trinomial.

$(y^2)^2 + 5(y^2) + 6 \quad = (a + 2)(a + 3)$

EXAMPLE Solving an equation

Solve the equation. $5y^3 - 35y^2 + 50y = 0$

Solution

$$5y^3 - 35y^2 + 50y = 0$$

$$5y(y^2 - 7y + 10) = 0$$

$$5y(y - 5)(y - 2) = 0$$

$$5y = 0 \quad \text{or} \quad y - 5 = 0 \quad \text{or} \quad y - 2 = 0$$

$$y = 0 \quad \text{or} \quad y = 5 \quad \text{or} \quad y = 2$$

The solutions are 0, 2, and 5.

Practice for section 6.7

- Factoring trinomials with common factors Q 9-20
- Factoring higher degree polynomials Q 33-46
- Solving an equation Q 51-68

7.6 Rational Equations and Formulas

- Solving Rational Equations
- Rational Expressions and Equations
- Graphical and Numerical Solutions
- Solving a Formula for a Variable
- Applications

If an equation contains one or more rational expressions, it is called a **rational equation**.

SOLVING BASIC RATIONAL EQUATIONS

The equations

$$\frac{a}{b} = \frac{c}{d} \quad \text{and} \quad ad = bc$$

are equivalent, provided that b and d are nonzero. Note that converting the first equation to the second equation is sometimes called *cross multiplying*.

EXAMPLE Solving rational equations

Solve each equation.

a. $\frac{9}{7} = \frac{4}{x}$

b. $\frac{6}{2x+1} = x$

Solution

a. $\frac{9}{7} = \frac{4}{x}$

$$9x = 28$$

$$x = \frac{28}{9}$$

The solution is $\frac{28}{9}$.

b. $\frac{6}{2x+1} = x$

$$\frac{6}{2x+1} = \frac{x}{1}$$

$$x(2x+1) = 6(1)$$

$$2x^2 + x = 6$$

$$2x^2 + x - 6 = 0$$

$$(2x-3)(x+2) = 0$$

$$2x - 3 = 0 \qquad x + 2 = 0$$

$$x = \frac{3}{2} \qquad x = -2$$

The solutions are $-2 \text{ and } \frac{3}{2}$.

STEPS FOR SOLVING A RATIONAL EQUATION

STEP 1: Find the LCD of the terms in the equation.

STEP 2: Multiply each side of the equation by the LCD.

STEP 3: Simplify each term.

STEP 4: Solve the resulting equation.

STEP 5: Check each answer in the *given* equation. Reject any value that makes a denominator equal 0.

EXAMPLE Identifying expressions and equations

Determine whether you are given an expression or an equation. If it is an expression, simplify it and then evaluate it for *x* = 4. If it is an equation, solve it.

a. $\dfrac{2x}{x+3} - 4 = \dfrac{x}{x+3}$

b. $\dfrac{x^2+3x}{x-2} - \dfrac{10}{x-2}$

Solution

a. There is an equal sign, so it is an equation.

$$(x+3)\frac{2x}{x+3} - 4(x+3) = (x+3)\frac{x}{x+3}$$

$$2x - 4x - 12 = x$$

$$-3x = 12$$

$$x = -4$$

The answer checks. Therefore the solution is −4.

EXAMPLE continued

b. $\dfrac{x^2+3x}{x-2}-\dfrac{10}{x-2}$

Solution

b. There is no equals sign, so it is an expression. The common denominator is $x - 2$, so we can add the numerators.

$$\frac{x^2+3x}{x-2}-\frac{10}{x-2} = \frac{x^2+3x-10}{x-2}$$

$$= \frac{(x-2)(x+5)}{x-2}$$

$$= x+5$$

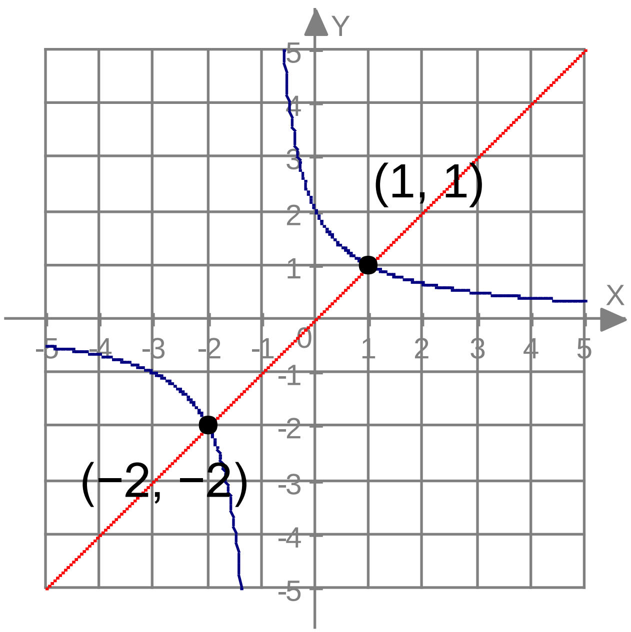
When $x = 4$, the expression evaluates $4 + 5 = 9$.

EXAMPLE Solving a rational equation graphically and numerically

Solve $\dfrac{2}{x+1} = x$ graphically and numerically.

Solution

Graph $y_1 = \dfrac{2}{x+1}$ and $y_2 = x$.



x	-3	-2	-1	0	1	2	3
$\dfrac{2}{x+1}$	-1	-2	—	2	1	$\dfrac{2}{3}$	$\dfrac{1}{2}$

The solutions are −2 and 1.

EXAMPLE continued

Solution

Numerical Solution $y_1 = \dfrac{2}{x+1}$ $y_2 = x.$

x	−3	−2	−1	0	1	2	3
$y_1 = \dfrac{2}{x+1}$	−1	−2	——	2	1	$\frac{2}{3}$	$\frac{1}{2}$
$y_2 = x$	−3	−2	−1	0	1	2	3

The solutions are −2 and 1.

EXAMPLE Solving a formula for a variable

Solve the equation $h = \frac{2A}{B + b}$ for A.

Solution

$$h = \frac{2A}{B + b}$$

$$h(B + b) = 2A$$

$$\frac{h(B + b)}{2} = A$$

Practice for section 7.6

- Solving rational equations Q 7-48
- Identifying expressions and equations Q 65-72
- Solving a rational equation graphically and numerically Q 77-82
- Solving a formula for a variable Q 89-100

7.7 Proportions and Variation

- Proportions
- Direct Variation
- Inverse Variation
- Joint Variation

A **proportion** is a statement (equation) that two ratios (fractions) are equal.

The following property is a convenient way to solve proportions:

$$\frac{a}{b} = \frac{c}{d} \quad \text{is equivalent to} \quad ad = bc,$$

provided $b \neq 0$ and $d \neq 0$.

EXAMPLE Calculating calories burned

On an elliptical machine, Francis can burn 370 calories in 25 minutes. If he increases his work time to 30 minutes, how many calories will he burn?

Solution

Let *x* be the equivalent amount of calories.

$$\frac{25}{370} = \frac{30}{x} \qquad \frac{\text{Minutes}}{\text{Calories}} = \frac{\text{Minutes}}{\text{Calories}}$$

$$25x = 11{,}100$$

$$x = 444$$

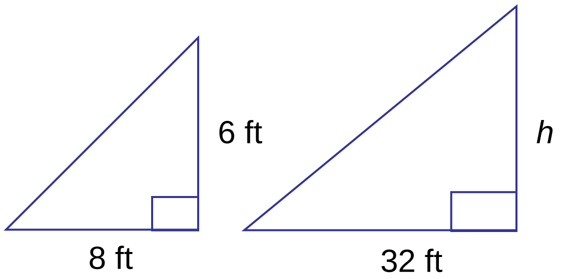
Thus, in 30 minutes, Francis will burn 444 calories.

EXAMPLE Calculating the height of a tree

A 6-foot tall person casts a shadow that is 8-foot long. If a nearby tree casts a 32-foot long shadow, estimate the height of the tree.

Solution

The triangles are similar because the measures of its corresponding angles are equal. Therefore corresponding sides are proportional.



$$\frac{h}{6} = \frac{32}{8} \qquad \frac{\text{Height}}{\text{Height}} = \frac{\text{Shadow length}}{\text{Shadow length}}$$

$$8h = 192$$

$$h = 24$$

The tree is 24 feet tall.

DIRECT VARIATION

Let x and y denote two quantities. Then y is **directly proportional** to x, or y **varies directly** with x, if there is a nonzero number k such that

$$y = kx.$$

The number k is called the **constant of proportionality**, or the **constant of variation**.

SOLVING A VARIATION APPLICATION

When solving a variation problem, the following steps can be used.

STEP 1: Write the general equation for the type of variation problem that you are solving.

STEP 2: Substitute given values in this equation so the constant of variation k is the only unknown value in the equation. Solve for k.

STEP 3: Substitute the value of k in the general equation in Step 1.

STEP 4: Use this equation to find the requested quantity.

EXAMPLE Solving a direct variation problem

Let *y* be directly proportional to *x*, or vary directly with *x*. Suppose *y* = 9 when *x* = 6. Find *y* when *x* = 13.

Solution

Step 1 The general equation is $y = kx$.

Step 2 Substitute 9 for *y* and 6 for *x* in $y = kx$. Solve for *k*.

$$y = kx$$
$$9 = 6k$$
$$\frac{9}{6} = k$$

Step 3 Replace *k* with 9/6 in the equation $y = 9x/6$.

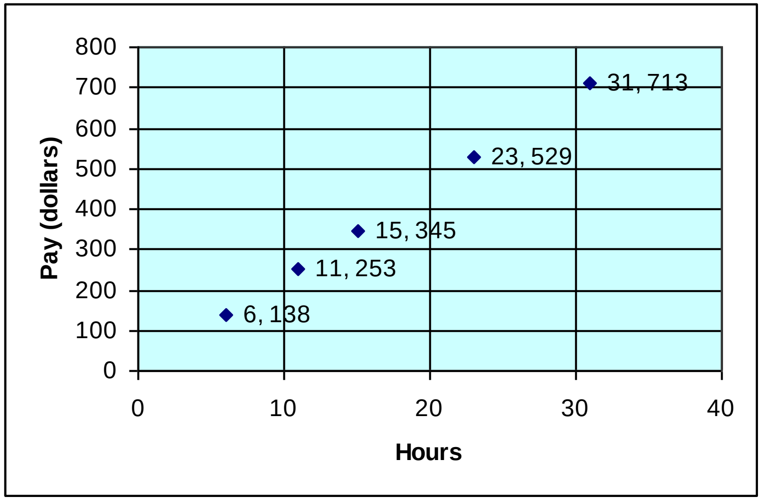
Step 4 To find *y*, let $x = 13$.

$$y = \frac{9}{6}x$$
$$y = \frac{9}{6}(13)$$
$$y = 19.5$$

EXAMPLE Modeling pay

The table lists the amount of pay for various hours worked.

Hours	6	11	15	23	31
Pay	$138	$253	$345	$529	$713



a. Find the constant of proportionality.

b. Predict the pay for 19 hours of work.

EXAMPLE continued

a. The slope of the line equals the proportionality, *k*. If we use the first and last data points (6, 138) and (31, 713), the slope is

$$k = \frac{713 - 138}{31 - 6} = 23$$

The amount of pay per hour is \$23. The graph of the line $y = 23x$, models the given graph.

b. To find the pay for 19 hours, substitute 19 for *x*.

$y = 23x$,

$y = 23(19)$

$y = 437$

19 hours of work would pay \$437.00

INVERSE VARIATION

Let x and y denote two quantities. Then y is **inversely proportional** to x, or y **varies inversely** with x, if there is a nonzero number k such that

$$y = \frac{k}{x}.$$

EXAMPLE Solving an inverse variation problem

Let *y* be inversely proportional to *x*, or vary inversely with *x*. Suppose *y* = 6 when *x* = 4. Find *y* when *x* = 8.

Solution

Step 1 The general equation is $y = k/x$.

Step 2 Substitute 6 for *y* and 4 for *x* in Solve for *k*.

$$y = \frac{k}{x}$$

$$6 = \frac{k}{4}$$

$$24 = k$$

Step 3 Replace *k* with 24 in the equation $y = k/x$.

Step 4 To find *y*, let *x* = 8.

$$y = \frac{k}{x}$$

$$y = \frac{24}{8}$$

$$y = 3$$

EXAMPLE Analyzing data

Determine whether the data in each table represent direct variation, inverse variation, or neither. For direct and inverse variation, find the equation.

a.

x	3	7	9	12
y	12	28	32	48

b.

x	5	10	12	15
y	12	6	5	4

c.

x	8	11	14	21
y	48	66	84	126

EXAMPLE continued

a.

x	3	7	9	12
y	12	28	32	48

Neither the product xy nor the ratio y/x are constant in the data in the table. Therefore there is neither direct variation nor indirect variation in this table.

b.

x	5	10	12	15
y	12	6	5	4

As x increases, y decreases. Because $xy = 60$ for each data point, the equation $y = 60/x$ models the data. This represents an inverse variation.

c.

x	8	11	14	21
y	48	66	84	126

The equation $y = 6x$ models the data. The data represents direct variation.

JOINT VARIATION

Let x, y, and z denote three quantities. Then z **varies jointly** with x and y if there is a nonzero number k such that

$$z = kxy.$$

EXAMPLE Finding the strength of a rectangular beam

The strength S of a rectangular beam varies jointly as its width w and the square of its thickness t. If a beam 5 inches wide and 2 inches thick supports 280 pounds, how much can a similar beam 4 inches wide and 3 inches thick support?

Solution

The strength of the beam is modeled by $S = kwt^2$.

$$280 = k \cdot 5 \cdot 2^2$$

$$k = \frac{280}{5 \cdot 4}$$

$$k = 14$$

EXAMPLE continued

Thus $S = 14wt^2$ models the strength of this type of beam. When $w = 4$ and $t = 3$, the beam can support

$$S = 14 \cdot 4 \cdot 3^2 = 504 \text{ pounds}$$

Practice for section 7.7

- Calculating calories burned Q 57
- Calculating the height of a tree Q 56
- Solving a direct variation problem Q 31
- Modeling pay none
- Solving an inverse variation problem Q 37
- Analyzing data Q 43-47
- Finding the strength of a rectangular beam Q 81

10.5 Equations Involving Radical Expressions

- Solving Radical Equations
- The Distance Formula
- Solving the Equation $x^n = k$

POWER RULE FOR SOLVING EQUATIONS

If each side of an equation is raised to the same positive integer power, then any solutions to the given equations are among the solutions to the new equation. That is, the solutions to the equation $a = b$ are among the solutions to $a^n = b^n$.

EXAMPLE Solving a radical equation symbolically

Solve $4\sqrt{2x+1} = 12$. Check your solution.

Solution

$$4\sqrt{2x+1} = 12$$
$$\sqrt{2x+1} = 3$$
$$\left(\sqrt{2x+1}\right)^2 = 3^2$$
$$2x+1 = 9$$
$$2x = 8$$
$$x = 4$$

Check:

$$4\sqrt{2(4)+1} = 12$$
$$4\sqrt{9} = 12$$
$$4(3) = 12$$
$$12 = 12$$

It checks.

SOLVING A RADICAL EQUATION

Step 1: Isolate a radical term on one side of the equation.

Step 2: Apply the power rule by raising each side of the equation to the power equal to the index of the isolated radical term.

Step 3: Solve the equation. If it still contains a radical, repeat Steps 1 and 2.

Step 4: Check your answers by substituting each result in the *given* equation.

EXAMPLE Isolating the radical term

Solve $\sqrt{6-x}-3=1$.

Solution

Step 1: To isolate the radical term, we add 3 to each side of the equation.

$$\sqrt{6-x}-3=1$$

$$\sqrt{6-x}=4$$

Step 2: Square each side.

$$\left(\sqrt{6-x}\right)^2=4^2$$

Step 3: Solve the resulting equation.

$$6-x=16$$

$$-x=10$$

$$x=-10$$

EXAMPLE continued

Step 4: Check your answer by substituting into the *given* equation.

$$\sqrt{6-x}-3=1$$

$$\sqrt{6-(-10)}-3=1$$

$$\sqrt{16}-3=1$$

$$4-3=1$$

$$1=1$$

Since this checks, the solution is $x = -10$.

EXAMPLE Solving a radical equation

Solve $\sqrt{3x-2} = x - 2$. Check your results and then solve the equation graphically.

Solution

Symbolic Solution

$$\sqrt{3x-2} = x-2$$

$$\left(\sqrt{3x-2}\right)^2 = (x-2)^2$$

$$3x - 2 = x^2 - 4x + 4$$

$$0 = x^2 - 7x + 6$$

$$0 = (x-6)(x-1)$$

$$x = 6 \quad \text{or} \quad x = 1$$

Check:

$$\sqrt{3x-2} = x-2$$

$$\sqrt{3(6)-2} = (6)-2$$

$$4 = 4$$

It checks.

$$\sqrt{3x-2} = x-2$$

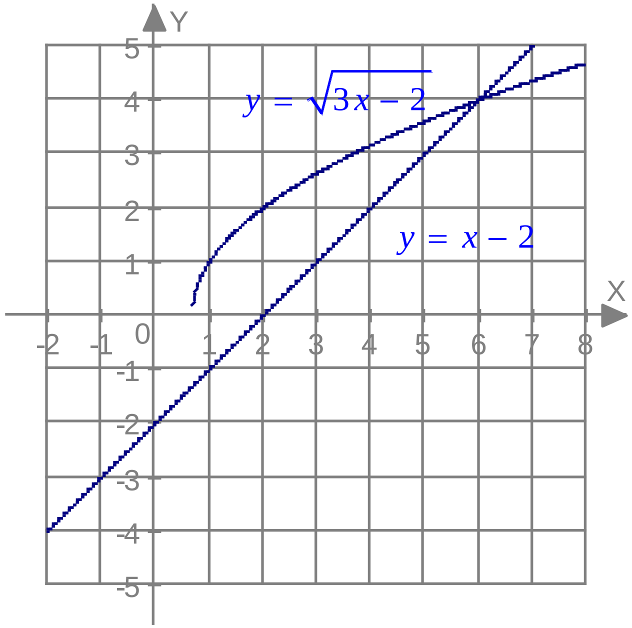
$$\sqrt{3(1)-2} = (1)-2$$

$$1 \neq -1$$

Thus 1 is an extraneous solution.

EXAMPLE continued

Graphical Solution



The solution 6 is supported graphically where the intersection is at (6, 4). *The graphical solution does not give an extraneous solution*.

EXAMPLE Solving an equation containing a cube root

Solve $\sqrt[3]{2x+6} = 4.$

Solution

Step 1: The cube root is already isolated, so we proceed to Step 2.

Step 2: Cube each side. $\left(\sqrt[3]{2x+6}\right)^3 = 4^3$

Step 3: Solve the resulting equation.

$$2x + 6 = 64$$

$$2x = 58$$

$$x = 29$$

EXAMPLE continued

Step 4: Check the answer by substituting into the given equation.

$$\sqrt[3]{2x+6}=4$$

$$\sqrt[3]{2(29)+6}=4$$

$$\sqrt[3]{64}=4$$

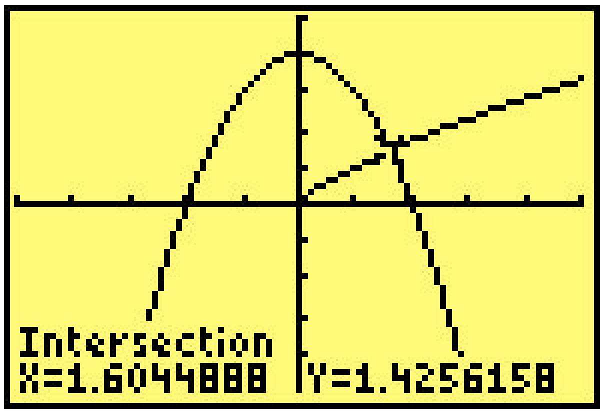
$$4=4$$

Since this checks, the solution is x = 29.

EXAMPLE

Solving an equation with rational exponents

Solve $x^{3/4} = 4 - x^2$ graphically. This equation would be difficult to solve symbolically, but an approximate solution can be found graphically.



EXAMPLE Applying the Pythagorean theorem

A 6ft ladder is placed against a garage with its base 3 ft from the building. How high above the ground is the top of the ladder?

Solution

$$c^2 = a^2 + b^2$$

$$6^2 = a^2 + 3^2$$

$$36 = a^2 + 9$$

$$27 = a^2$$

$$\sqrt{27} = a$$

$$3\sqrt{3} = a$$

The ladder is 5.2 ft above ground.

DISTANCE FORMULA

The distance d between the points (x_1, y_1) and (x_2, y_2) in the xy-plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE Finding distance between points

Find the distance between the points (−1, 2) and (6, 4).

Solution

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

$$d = \sqrt{(6-(-1))^2 + (4-2)^2}$$

$$d = \sqrt{49 + 4}$$

$$d = \sqrt{53}$$

$$d \approx 7.28$$

SOLVING THE EQUATION $x^n = k$

Take the *n*th root of each side of $x^n = k$ to obtain $\sqrt[n]{x^n} = \sqrt[n]{k}$.

1. If n is odd, then $\sqrt[n]{x^n} = x$ and the equation becomes $x = \sqrt[n]{k}$.

2. If n is *even* and $k > 0$, then $\sqrt[n]{x^n} = |x|$ and the equation becomes $|x| = \sqrt[n]{k}$. (If $k < 0$, there are no real solutions.)

EXAMPLE Solving the equation $x^n = k$

Solve each equation.

a. $x^3 = -216$ b. $x^2 = 17$ c. $3(x + 4)^4 = 48$

Solution

a. $x^3 = -216$

$$\sqrt[3]{x^3} = \sqrt[3]{-216}$$

$$x = -6$$

b. $x^2 = 17$

$$\sqrt{x^2} = \sqrt{17}$$

or

$$|x| = \sqrt{17}$$

$$x = \pm\sqrt{17}$$

EXAMPLE continued

c. $3(x + 4)^4 = 48$

$$(x + 4)^4 = 16$$

$$\sqrt[4]{(x+4)} = \sqrt[4]{16}$$

$$|x + 4| = 2$$

$$x + 4 = -2 \qquad x + 4 = 2$$

$$x = -6 \qquad x = -2$$

EXAMPLE Modeling volume of a sphere

The formula for the volume (V) of a sphere with a radius (r), is given by $V = \frac{4}{3}\pi r^3$. Solve for r.

Solution

$$V = \frac{4}{3}\pi r^3$$

$$\frac{3V}{4\pi} = r^3$$

$$\sqrt[3]{\frac{3V}{4\pi}} = r$$

Practice for section 10.5

- Solving a radical equation symbolically Q 21-22
- Isolating the radical term Q 23-26
- Solving a radical equation Q 27,35,37,41
- Solving an equation containing a cube root Q 31
- Solving an equation with rational exponents Q 73
- Applying the Pythagorean theorem Q 123
- Finding distance between points Q 109
- Solving the equation $x^n = k$ Q 47, 55, 67
- Modeling volume of a sphere none

11.3 Quadratic Equations

- Basics of Quadratic Equations
- The Square Root Property
- Completing the Square
- Solving an Equation for a Variable
- Applications of Quadratic Equations

Quadratic Functions

Any quadratic function f can be represented by $f(x) = ax^2 + bx + c$ with $a \neq 0$.

Examples:

$$f(x) = 2x^2 - 1,\ g(x) = -\frac{1}{3}x^2 + 2x,\ \text{and}\ h(x) = x^2 + 2x - 1$$

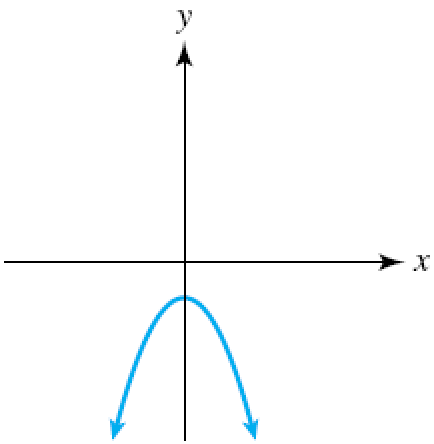
QUADRATIC EQUATION

A **quadratic equation** is an equation that can be written as

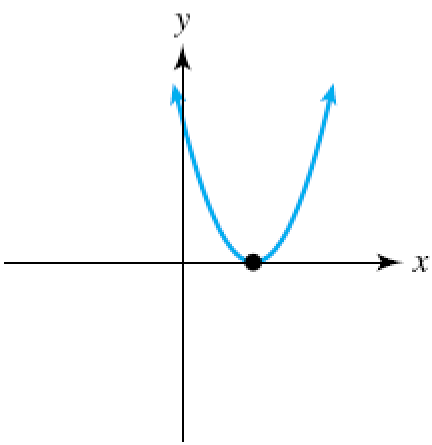
$$ax^2 + bx + c = 0,$$

where a, b, and c are constants with $a \neq 0$.

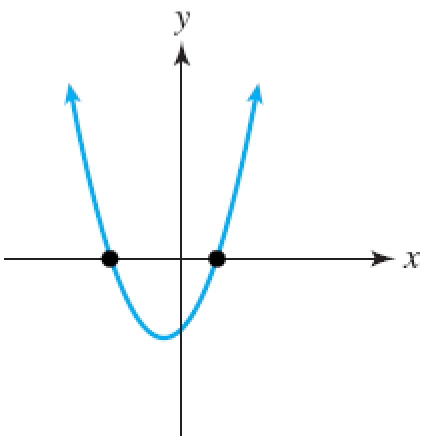
The different types of solutions to a quadratic equation.



(a) No x-intercepts



(b) One x-intercept



(c) Two x-intercepts

EXAMPLE Solving quadratic equations

Solve each quadratic equation. Support your results numerically and graphically.

a. $3x^2 + 2 = 0$ b. $x^2 + 9 = -6x$ c. $x^2 + 2x - 8 = 0$

Solution

Symbolic:

$$3x^2 + 2 = 0$$

$$3x^2 = -2$$

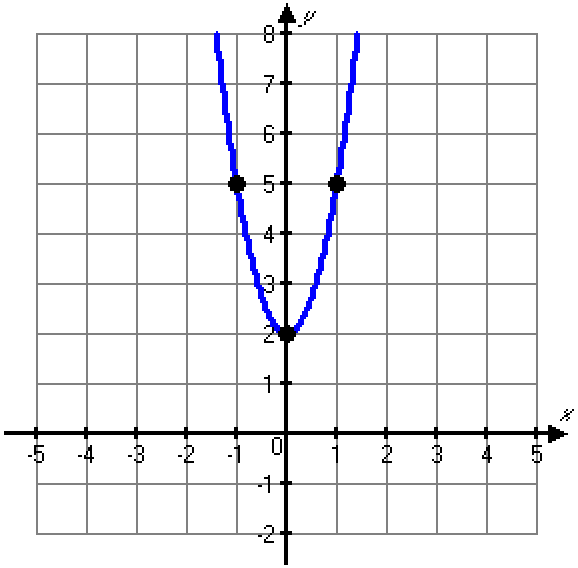
$$x^2 = -\frac{2}{3}$$

The equation has no real solutions because $x^2 \geq 0$ for all real numbers x.

Numerical:

x	y
–1	5
0	2
1	5

Graphical:



EXAMPLE Solving quadratic equations

Solve each quadratic equation. Support your results numerically and graphically.

a. $3x^2 + 2 = 0$

b. $x^2 + 9 = -6x$

c. $x^2 + 2x - 8 = 0$

Solution

Symbolic:

$$x^2 + 9 = -6x$$

$$x^2 + 6x + 9 = 0$$

$$(x + 3)(x + 3) = 0$$

$$x + 3 = 0 \quad \text{or} \quad x + 3 = 0$$

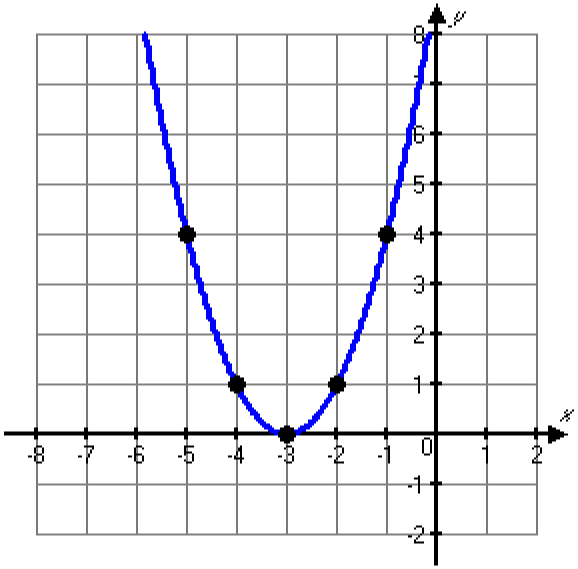
$$x = -3 \quad \text{or} \quad x = -3$$

The equation has one real solution.

Numerical:

x	y
–5	4
–4	1
–3	0
–2	1
–1	4

Graphical:



EXAMPLE Solving quadratic equations

Solve each quadratic equation. Support your results numerically and graphically.

a. $3x^2 + 2 = 0$ b. $x^2 + 9 = -6x$ c. $x^2 + 2x - 8 = 0$

Solution

Symbolic:

$$x^2 + 2x - 8 = 0$$

$$(x - 2)(x + 4) = 0$$

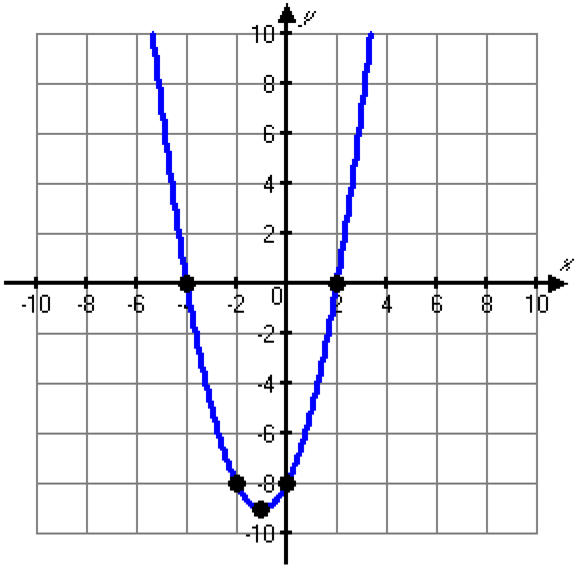
$$x - 2 = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = 2 \quad \text{or} \quad x = -4$$

Numerical:

x	y
–4	0
–2	–8
–1	–9
0	–8
2	0

Graphical:



The equation has two real solutions.

The **square root property** is used to solve quadratic equations that have no *x*-terms.

SQUARE ROOT PROPERTY

Let k be a nonnegative number. Then the solutions to the equation

$$x^2 = k$$

are given by $x = \pm\sqrt{k}$. If $k < 0$, then this equation has no real solutions.

EXAMPLE Using the square root property

Solve each equation.

a. $x^2 = 10$

b. $25x^2 - 16 = 0$

c. $(x-3)^2 = 16$

Solution

a. $x^2 = 10$

$$\sqrt{x^2} = \sqrt{10}$$

$$x = \pm\sqrt{10}$$

b. $25x^2 - 16 = 0$

$$25x^2 = 16$$

$$x^2 = \frac{16}{25}$$

$$x = \pm\sqrt{\frac{16}{25}}$$

$$x = \pm\frac{4}{5}$$

c. $(x-3)^2 = 16$

$$(x-3) = \pm\sqrt{16}$$

$$x - 3 = \pm 4$$

$$x = 3 \pm 4$$

$$x = -1 \quad \text{or} \quad 7$$

Real World Connection: If an object is dropped from a height of *h* feet, its distance *d* above the ground after *t* seconds is given by

$$d(t) = h - 16t^2$$

EXAMPLE Modeling a falling object

A toy falls 40 feet from a window. How long does the toy take to hit the ground?

Solution

$$d(t) = h - 16t^2$$

$$d(t) = 40 - 16t^2$$

$$40 - 16t^2 = 0$$

$$-16t^2 = -40$$

$$t^2 = \frac{40}{16}$$

$$t = \pm\sqrt{\frac{40}{16}} = \frac{\sqrt{4 \cdot 10}}{\sqrt{16}} = \frac{2\sqrt{10}}{4} = \frac{\sqrt{10}}{2} \approx 1.6 \text{ sec.}$$

The method of completing the square can be used to solve quadratic equations.

EXAMPLE Creating a perfect square trinomial

Find the term that should be added to $x^2 - 8x$ to form a perfect square trinomial.

Solution

Coefficient of *x*-term is –8, so we let *b* = –8. To complete the square we divide by 2 and then square the result.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = 16$$

$$x^2 - 8x + 16 = (x - 4)^2$$

EXAMPLE

Completing the square when the leading coefficient is 1

Solve the equation $x^2 - 8x + 13 = 0$

Solution

Write the equation in $x^2 + bx = d$ form.

$$x^2 - 8x = -13$$

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = 16$$

$$x^2 - 8x + 16 = -13 + 16$$

$$(x-4)^2 = 3$$

$$x - 4 = \pm\sqrt{3}$$

$$x = 4 \pm \sqrt{3}$$

$$x \approx 5.73 \text{ or } 2.27$$

EXAMPLE

Completing the square when the leading coefficient is not 1

Solve the equation $0 = 2x^2 + 8x + 7$

Solution

Write the equation in $x^2 + bx = d$ form.

$$2x^2 + 8x = -7$$

$$x^2 + 4x = \frac{-7}{2}$$

$$\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 4$$

$$x^2 + 4x + 4 = -\frac{7}{2} + 4$$

$$(x+2)^2 = \frac{1}{2}$$

$$x + 2 = \pm\sqrt{\frac{1}{2}}$$

$$x + 2 = \pm\frac{1}{\sqrt{2}}$$

$$x + 2 = \pm\frac{\sqrt{2}}{2}$$

$$x = -2 \pm \frac{\sqrt{2}}{2}$$

$$x \approx -1.29 \quad \text{or} \quad -2.71$$

EXAMPLE **Solving equations for variables**

Solve the equation for the specified variable. $w = 16x^2$ for x

Solution

$$w = 16x^2$$

$$x^2 = \frac{w}{16}$$

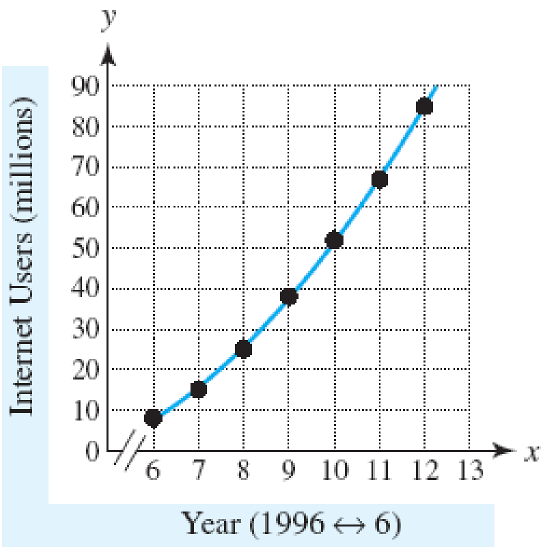
$$x = \pm\sqrt{\frac{w}{16}}$$

$$x = \pm\frac{\sqrt{w}}{4}$$

EXAMPLE Modeling numbers of Internet users

Use of the Internet in Western Europe has increased dramatically. The figure shows a scatter plot of online users in Western Europe, together with a graph of a function *f* that models the data. The function *f* is given by: $f(x) = 0.976x^2 - 4.643x + 0.238$ where the output is in millions of users. In this formula *x* = 6 corresponds to 1996, *x* = 7 to 1997, and so on, until *x* = 12 represents 2002.

a. Evaluate *f*(10) and interpret the result.
b. Graph *f* and estimate the year when the number of Internet users reached 85 million.
c. Solve part (b) numerically.



EXAMPLE Modeling numbers of Internet users

Solution

$$f(x) = 0.976x^2 - 4.643x + 0.238$$

a. Evaluate *f*(10) and interpret the result.

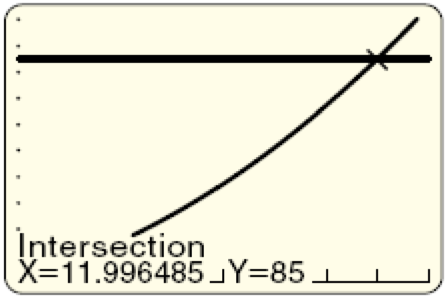
$$f(x) = 0.976(10)^2 - 4.643(10) + .0238$$
$$= 51.4$$

Because *x* = 10 corresponds to 2000, there were about 51.4 million users in 2000.

b. Graph *f* and estimate the year when the number of Internet users reached 85 million.

c. Solve part (b) numerically.

[5, 13, 1] by [0, 100, 10]



X	Y_1	Y_2
9	37.507	85
10	51.408	85
11	67.261	85
12	85.066	85
13	104.82	85
14	126.53	85
15	150.19	85

X=12

Practice for section 11.3

- Solving quadratic equations Q 29-40
- Using the square root property Q 51-62
- Modeling a falling object Q 115
- Creating a perfect square trinomial Q 67-70
- Completing the square when the leading coefficient is 1 Q 72-76
- Completing the square when the leading coefficient is not 1 Q 81-86
- Solving equations for variables Q 107-108
- Modeling numbers of Internet users Q 124

11.4

The Quadratic Formula

- Solving Quadratic Equations
- The Discriminant
- Quadratic Equations Having Complex Solutions

QUADRATIC FORMULA

The solutions to $ax^2 + bx + c = 0$ with $a \neq 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE

Solving a quadratic equation having two solutions

Solve the equation $4x^2 + 3x – 8 = 0$. Support your results graphically.

Solution

Symbolic Solution

Let $a = 4$, $b = 3$ and $c = -8$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(4)(-8)}}{2(4)}$$

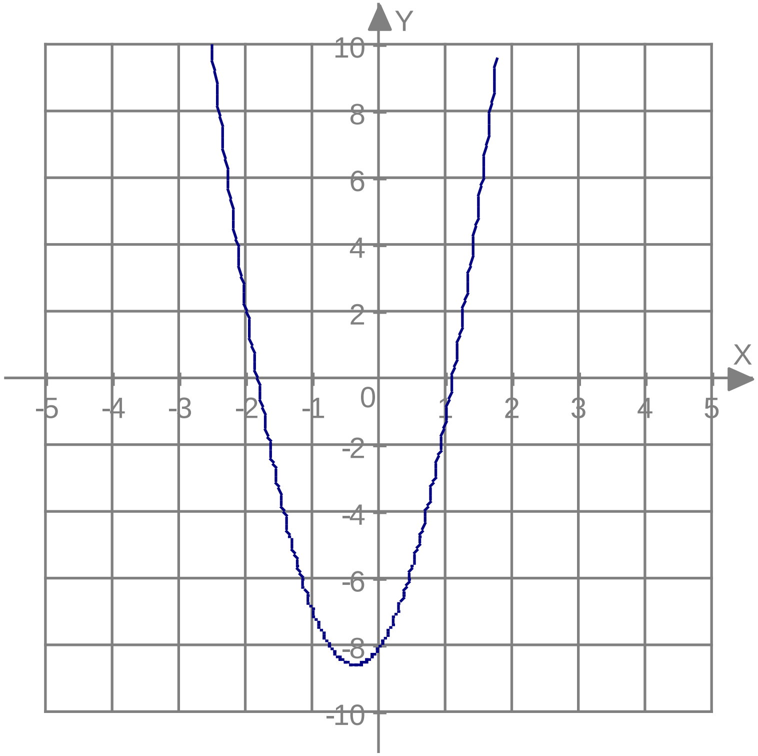
$$x = \frac{-3 \pm \sqrt{137}}{8}$$

$$x = \frac{-3 + \sqrt{137}}{8} \quad \text{or} \quad x = \frac{-3 - \sqrt{137}}{8}$$

$$x \approx 1.1 \quad \text{or} \quad x \approx -1.8$$

EXAMPLE continued

Graphical Solution



$$y = 4x^2 + 3x - 8$$

EXAMPLE **Solving a quadratic equation having one solution**

Solve the equation $3x^2 - 6x + 3 = 0$. Support your result graphically.

Solution

Symbolic Solution

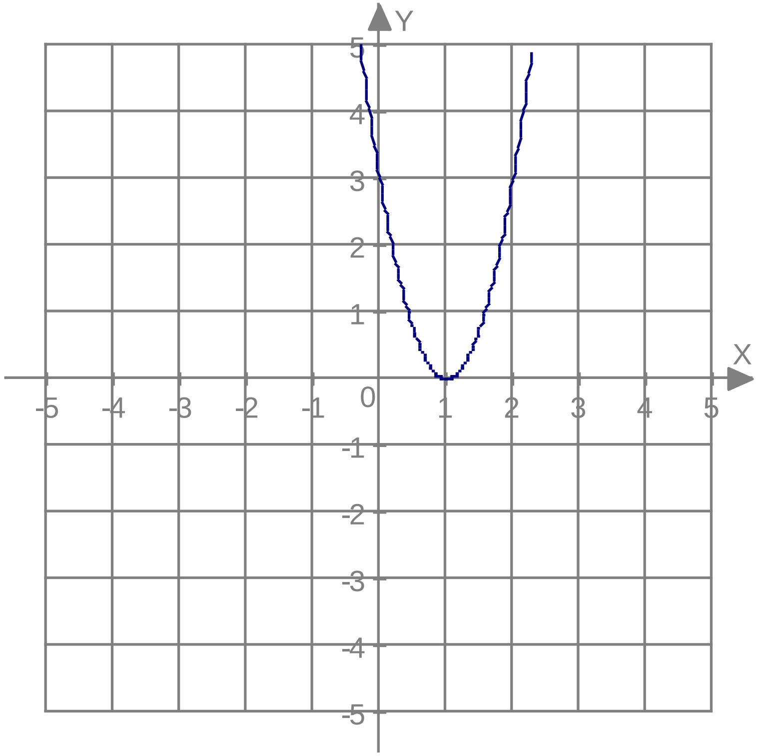
Let $a = 3$, $b = -6$ and $c = 3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(3)}}{2(3)} \qquad x = \frac{6 \pm \sqrt{0}}{6} \qquad x = 1$$

EXAMPLE continued

Graphical Solution



$$y = 3x^2 - 6x + 3$$

EXAMPLE Solving a quadratic equation having no real solutions

Solve the equation $2x^2 + 4x + 5 = 0$. Support your result graphically.

Solution

Symbolic Solution

Let $a = 2$, $b = 4$ and $c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

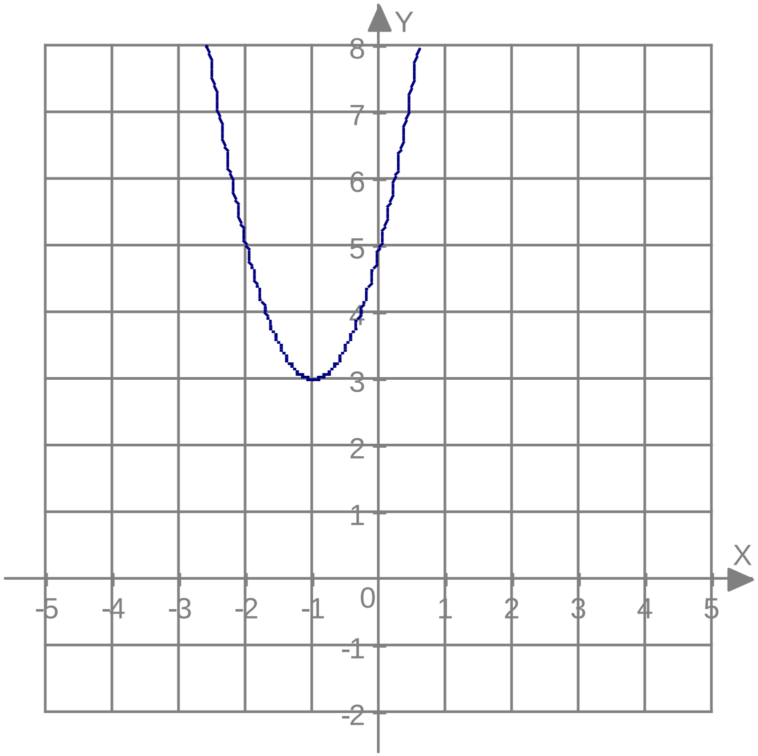
$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(5)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{-24}}{4}$$

There are no real solutions for this equation because $\sqrt{-24}$ is not a real number.

EXAMPLE continued

Graphical Solution



$$y = 2x^2 + 4x + 5$$

THE DISCRIMINANT AND QUADRATIC EQUATIONS

To determine the number of solutions to the quadratic equation $ax^2 + bx + c = 0$, evaluate the discriminant $b^2 - 4ac$.

1. If $b^2 - 4ac > 0$, there are two real solutions.

2. If $b^2 - 4ac = 0$, there is one real solution.

3. If $b^2 - 4ac < 0$, there are no real solutions; there are two complex solutions.

EXAMPLE Using the discriminant

Use the discriminant to determine the number of solutions to $-2x^2 + 5x = 3$. Then solve the equation using the quadratic formula.

Solution

$$-2x^2 + 5x - 3 = 0$$

Let $a = -2$, $b = 5$ and $c = -3$.

$$b^2 - 4ac$$

$$= (5)^2 - 4(-2)(-3) = 1$$

Thus, there are two solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-5 \pm \sqrt{1}}{2(-2)}$$

$$x = \frac{-4}{-4} \quad \text{or} \quad x = \frac{-6}{-4}$$

$$x = 1 \qquad x = 1.5$$

THE EQUATION $x^2 + k = 0$

If $k > 0$, the solution to $x^2 + k = 0$ are given by $x = \pm i\sqrt{k}$.

EXAMPLE Solving a quadratic equation having complex solutions

Solve $x^2 + 17 = 0$.

Solution

The solutions are $\pm i\sqrt{17}$.

That is, $x = i\sqrt{17} \text{ or } -i\sqrt{17}$.

EXAMPLE

Solving a quadratic equation having complex solutions

Solve $3x^2 - 7x + 5 = 0$. Write your answer in standard form: $a + bi$.

Solution

Let $a = 3$, $b = -7$ and $c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(5)}}{2(3)}$$

$$x = \frac{7 \pm i\sqrt{11}}{6}$$

$$x = \frac{7 \pm \sqrt{-11}}{6}$$

$$x = \frac{7}{6} + i\frac{\sqrt{11}}{6} \quad \text{and} \quad x = \frac{7}{6} - i\frac{\sqrt{11}}{6}$$

EXAMPLE Solving a quadratic equation having complex solutions

Solve $-\frac{2x^2}{5} - 3 = -2x$. Write your answer in standard form: *a* + *bi*.

Solution

Begin by adding 2*x* to each side of the equation and then multiply by 5 to clear fractions.

$$-2x^2 + 10x - 15 = 0$$

Let *a* = −2, *b* = 10 and *c* = −15.

EXAMPLE continued

Let $a = -2$, $b = 10$ and $c = -15$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(-2)(-15)}}{2(-2)}$$

$$x = \frac{-10 \pm \sqrt{-20}}{-4}$$

$$x = \frac{-10 \pm 2i\sqrt{5}}{-4}$$

$$x = \frac{5}{2} \pm i\frac{\sqrt{5}}{2}$$

EXAMPLE Completing the square to find complex solutions

Solve $x(x-4)=-5$ by completing the square.

Solution

After applying the distributive property, the equation becomes $x^2-4x=-5$.

Since *b* = −4 ,add $\left(\frac{-4}{2}\right)^2=4$ to each side of the equation.

$$x^2-4x+4=-5+4$$
$$(x-2)^2=-1$$
$$x-2=\pm\sqrt{-1}$$
$$x-2=\pm i$$
$$x=2\pm i$$

The solutions are 2 + *i* and 2 − *i*.

Practice for section 11.4

- Solving a quadratic equation having two solutions Q 9-10
- Solving a quadratic equation having one solution Q 11
- Solving a quadratic equation having no real solutions Q 12
- Using the discriminant Q 37-41 a,b's
- Solving a quadratic equation having complex solutions Q 55-86 (not 75)
- Completing the square to find complex solutions Q75

MTH 209 End of week 3

- You again have the answers to those problems not assigned
- Practice is SOOO important in this course.
- Work as much as you can with MyMathLab, the materials in the text, and on my Webpage.
- Do everything you can scrape time up for, first the hardest topics then the easiest.
- You are building a skill like typing, skiing, playing a game, solving puzzles.
- NEXT TIME: Nonlinear Functions, Sequences and Series